Delta method | a way to do approximation [expectation & variance]

motivation: $E(X) = \mu_x$ $Var(X) = \sigma_x^2$ and $Y = g(X)$

How to find $E(Y)$ & $Var(Y)$

- idea: use defn of Expectation

$$E(Y) = E(g(X)) = \begin{cases} \sum g(x) P(X=x) & \text{, if } X \text{ discrete} \\ \int_{-\infty}^{+\infty} g(x) f(x) dx & \text{, if } X \text{ continous} \end{cases}$$

↳ PMF/PDF of X is unknown, what's known of the X dist are only $E(X)$ and $Var(X)$

- Approximate (by delta method)

approx. Mean

- By Taylor expansion: (approx $g(x)$ by a polynomial fun. of X)

$$Y = g(X) = g(\mu_x) + \underbrace{(X-\mu_x) g'(\mu_x)}_{\text{first order}} + \underbrace{\frac{(X-\mu_x)^2}{2} g''(\mu_x)}_{\text{second order}} + \dots$$

↳ $g(\cdot)$: $g(\mu_x)$, $g'(\mu_x)$, $g''(\mu_x)$ constant which we know

when take Expectation, we have $E(X-\mu_x)$, $E(X-\mu_x)^2$

$$E(Y) = g(\mu_x) + E(X-\mu_x) \cdot g'(\mu_x) + E(X-\mu_x)^2 \frac{g''(\mu_x)}{2} + \dots$$

$$\approx g(\mu_x) + \underbrace{E(X) - \mu_x}_{0} \cdot g'(\mu_x) + \{\underbrace{Var(X-\mu_x)}_{= Var X = \sigma_x^2} + \underbrace{[E(X-\mu_x)]^2}_{0}\} \frac{g''(\mu_x)}{2}$$

only keep first 3 term

$$E(Y) \approx g(\mu_x) + \frac{g''(\mu_x)}{2} \sigma_x^2$$

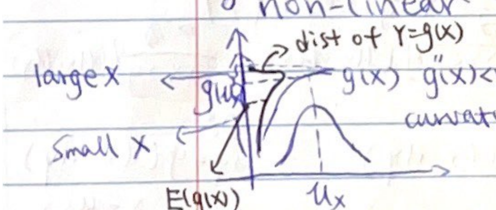
- special case: linear transformation

$g(x) = ax + b$, then $g''(x) = 0$ $\forall x \in \mathbb{R}$ $\Rightarrow E(Y) = g(\mu_x) + 0 = g(EX)$

$\therefore$ swap E & g $E(g(X)) = g(EX)$

$E(aX+b) = aEX + b = g(EX)$

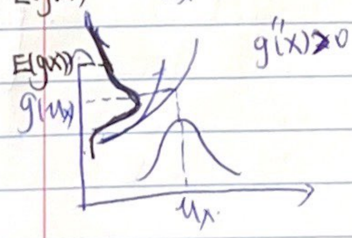
- non-linear transformation



~~E(Y) = E(g(X)) = g(u_x)~~

dist of Y not symmetric.

$$g(\mu_x) > E(g(X)) = g(\mu_x) + \underbrace{\frac{g''(\mu_x)}{2}}_{<0} \sigma_x^2$$ correction term



$E(g(x)) > g(\mu_x)$

$\therefore$ basically, we're using ~~E(X)~~ =

$E(Y) = E(g(X)) \approx \underbrace{g(\mu_x)}_{\text{known knowledge}}$ + correction term

to approx. $E(Y)$

approx. var Y

$Y = g(X) = g(\mu_X) + (X-\mu_X)\, g'(\mu_X) + \frac{(X-\mu_X)^2}{2} g''(\mu_X)$

$var(Y) = Var(g(X)) = [g'(\mu_X)]^2 \, var X + \ldots$

$\approx [g'(\mu_X)]^2 \sigma_X^2$ [drop other terms, since have no info abt high ~ terms

To summarize, based on ~~ty~~ taylor expansion,

Delta method gives:

$$\begin{cases} E(Y) = E(g(X)) = g(\mu_X) + \frac{g''(\mu_X)}{2} \sigma_X^2 \\ Var(Y) = Var(g(X)) = [g'(\mu_X)]^2 \cdot \sigma_X^2 \end{cases}$$

prime, not double prime

Example $X \sim Unif[10, 20]$. Find the exact & approximate mean & var of $Y = \frac{1}{X}$.

- exact: $E(Y) = E(\frac{1}{X}) = \int_{10}^{20} \frac{1}{X} \cdot \frac{1}{20-10} dx = \frac{1}{10} \log X \Big|_{10}^{20} = \frac{1}{10}(\log \frac{20}{10}) = \frac{1}{10}\log 2$

$= 0.0693$

$var(Y) = var(\frac{1}{X}) = E[(\frac{1}{X})^2] - (E\frac{1}{X})^2$

$= \int_{10}^{20} \frac{1}{X^2} \frac{1}{20-10} dx - [\frac{1}{10}\log 2]^2$

$= \frac{1}{10}(-\frac{1}{X})\Big|_{10}^{20} - \frac{1}{10}\log^2$

$= -\frac{1}{200} + \frac{1}{100} - \frac{1}{10}\log^2$

$= \frac{1}{200} - \frac{1}{10}\log^2 \approx 0.000195$

- Approx.: $E(Y) = E(g(X)) = g(\mu_X) + \frac{g''(\mu_X)}{2} \sigma_X^2$

$\mu_X = E(X) = 15$, $\sigma_X^2 = Var(X) = \frac{25}{3}$, $g(X) = \frac{1}{X}$, $g'(X) = -\frac{1}{X^2}$, $g''(X) = \frac{2}{X^3}$

$E(Y) = \frac{1}{15} + \frac{2}{(15)^3 \cdot 2} \cdot \frac{25}{3} = 0.0691$ $\quad var(Y) = (\frac{1}{15})^2 \cdot \frac{25}{3} = 0.000165$

Bivariate Taylor expansion: $g(X, Y)$ expand it at (μ_X, μ_Y)

$$g(X,Y) \approx g(\mu_X, \mu_Y) + (X-\mu_X, Y-\mu_Y)^T \begin{pmatrix} \frac{\partial g}{\partial x}(\mu_X, \mu_Y) \\ \frac{\partial g}{\partial y}(\mu_X, \mu_Y) \end{pmatrix}$$

$$+ \frac{1}{2}(X-\mu_X, Y-\mu_Y) \begin{bmatrix} \frac{\partial^2}{\partial x^2} g(\mu_X, \mu_Y) & \frac{\partial^2 g}{\partial x \partial y}(\mu_X, \mu_Y) \\ \frac{\partial^2}{\partial x \partial y} g(\mu_X, \mu_Y) & \frac{\partial^2}{\partial y^2} g(\mu_X, \mu_Y) \end{bmatrix} \cdot \begin{pmatrix} X-\mu_X \\ Y-\mu_Y \end{pmatrix}$$