convergence of MGF implies ~~conv~~ convergence of CDF.

Continuity Theorem

Let $X_1, X_2 \dots X_n$ be a sequence of R.V.s with CDFs $F_1(\cdot), F_2(\cdot) \dots$ and $M_{X_1}(\cdot), M_{X_2}(\cdot) \dots$ Suppose that:

$$M_{X_n}(t) \longrightarrow M_X(t) \text{ as } n \to \infty \text{ for all } t \in \mathbb{R}$$

where X is another R.V. with CDF $F(\cdot)$ and MGF $M_X(\cdot)$

Then $F_n(x) \longrightarrow F(x)$ as $n \to \infty$ for all x at which F is continuous

example

poisson(λ) converges to a normal dist as $\lambda \to \infty$. as $n \to \infty$

- Let $\lambda_1 < \lambda_2 < \lambda_3 \dots < \lambda_n < \dots$ be an increasing sequence $\lambda_n \to \infty$
- Let $X_n \sim pois(\lambda_n)$ $E(X_n) = \lambda_n$ $Var(X_n) = \lambda_n$
- MGF of pois(λ_n) $M_{X_n}(t) = e^{\lambda_n(e^t - 1)}$ $t \in \mathbb{R}$
- standardize X_n as:

otherwise var $\to \infty$

$$Z_n = \frac{X_n - \lambda_n}{\sqrt{\lambda_n}}$$

$\to$ Now, we want to show Z_n converges to $N(0,1)$ as $\lambda_n \to \infty$ ($n \to \infty$)

X_n is pois. Z_n is a function of X_n, then

$$Z_n = -\frac{\cancel{\lambda_n}}{\sqrt{\lambda_n}} + \left(\frac{1}{\sqrt{\lambda_n}}\right) \cdot X_n$$

$$M_{Z_n}(t) = e^{(-\cancel{\sqrt{\lambda_n}})t} \cdot M_{X_n}\left(\frac{1}{\sqrt{\lambda_n}} t\right) \text{ [by MGF property ①]}$$

$$= e^{(-\cancel{\sqrt{\lambda_n}}t)} \cdot e^{\lambda_n\left(e^{\frac{t}{\sqrt{\lambda_n}}} - 1\right)} \quad ①$$

What to show $M_{Z_n}(t)$ is the MGF of $N(0,1)$, which is $e^{\frac{t^2}{2}}$ ② $M_Z(t) = e^{\frac{t^2}{2}}$

try to p.f. ① = ② $\Rightarrow \left(-\cancel{\sqrt{\lambda_n}}t + \lambda_n\left(e^{\frac{t}{\sqrt{\lambda_n}}} - 1\right)\right) = \frac{1}{2}t^2$ as $n \to \infty$

using taylor expansion $e^x = e^0 + xe^0 + \frac{x^2}{2}e^0 + \dots$ $[(e^x)' = e^x]$

$$-\sqrt{\lambda_n}\, t + \lambda_n\Big(\underbrace{1 + \frac{t}{\sqrt{\lambda_n}} + \frac{t^2}{2\lambda_n} + \frac{1}{3!}\left(\frac{t}{\sqrt{\lambda_n}}\right)^3}_{\text{taylor expansion}} - 1\Big) = -\sqrt{\lambda_n}\, t + \sqrt{\lambda_n}\, t + \frac{t^2}{2} + \frac{1}{3!}\frac{t^3}{\sqrt{\lambda_n}}$$

$$= \frac{t^2}{2} + \frac{1}{3!}\frac{t^3}{\sqrt{\lambda_n}} + o\left(\frac{1}{\sqrt{\lambda_n}}\right)$$

$$\boxed{a_n = o(b_n) \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = 0}$$

a_n goes to 0 faster than b_n does

$\frac{1}{n^2} = o\left(\frac{1}{n}\right)$. $\frac{1}{\lambda_n^2} = o\left(\frac{1}{\sqrt{\lambda_n}}\right)$

$\therefore \lambda_n \to \infty$, $\frac{t^2}{2} + \lim_{\lambda_n \to \infty}\left(\frac{1}{3!}\frac{1}{\sqrt{\lambda_n}}t^3 + o\left(\frac{1}{\sqrt{\lambda_n}}\right)\right)$

$$= \frac{t^2}{2}$$

$\therefore$ we've shown $\log M_{Z_n}(t) \xrightarrow{n\to\infty} \log M_Z(t) \Rightarrow M_{Z_n}(t) \xrightarrow{n\to\infty} M_Z(t) \xrightarrow{\text{continuity Thm}} Z_n \xrightarrow{d} Z(0,1)$.

$\therefore$ standardized poisson(λ) RV conv. in dist to $N(0,1)$ as $\lambda \to \infty$ ∎

Central Limit Theorem

Let $X_1, X_2 \dots X_n$ be iid R.V.s with mean μ and Var. $\sigma^2 < \infty$, and MGF$(\cdot)$ defined in the neighborhood of 0.

Let $S_n = \sum_{i=1}^{n} X_i$; $\bar{X}_n = \frac{1}{n} S_n$ ☆

↳ Standardize S_n:

$$\begin{cases} E(S_n) = n\mu \\ Var(S_n) = n\sigma^2 \\ E(\bar{X}_n) = E(\frac{S_n}{n}) = \mu \\ Var(\bar{X}_n) = Var(\frac{S_n}{n}) = \frac{\sigma^2}{n} \end{cases} \qquad \frac{S_n - E(S_n)}{\sqrt{Var(S_n)}} = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{S_n - n\mu}{\sqrt{n}\,\sigma} \quad ①$$

- do the experiment n times $(X_1, X_2 \dots X_n)$ → take the average $\bar{X}_n = \frac{1}{n} S_n = \frac{1}{n} \sum_{i=1}^{n} X_i$
  → $\bar{X}_n$ becomes stable $Var(\bar{X}_n) = \frac{\sigma^2}{n \uparrow}$
- $E(\bar{X}_n)$ $Var(\bar{X}_n)$ always use capital $\bar{X}_n$. to represent R.V.s

↳ Standardize $\bar{X}_n$:

$$\frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}} = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{S_n/n - \mu}{\sigma/\sqrt{n}} = \frac{S_n - n\mu}{\sqrt{n}\,\sigma} \quad ②$$

Standardize Sum ① = standardize average ②

CLT statement

$$\boxed{\frac{S_n - n\mu}{\sqrt{n}\,\sigma} = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1) \text{ as } n \to \infty}$$

↳ # of samples in one realization

☆ $\sigma^2 < \infty$

☆ No matter what dist you have, with variance $< \infty$ (finite var.)
The standardized average will follow Normal $N(0,1)$ ☆

realization ← ① → collect n: $X_1 \dots X_n$ → calculate $\bar{X}_n$

imagination: ② → collect n: another $X_1 \dots X_n$ → calculate $\bar{X}_n$
⋮
Ⓚ repeat to get the K^{th} $\bar{X}_n$

⇒ end up with dist of $\bar{X}_n$.

☆ in practice, we usually only have one realization of the experiment, say ①, so we only have one $\bar{X}_n$, what we do is to use imagination to see potential behavior of collective data (which not being collected) had they collected. [See beyond observed]
interests: How K $\bar{X}_n$ collectively looks like?
what we have: 1 $\bar{X}_n$ from n samples / (K−1) → imagination.

Proof CLT

Let $Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}}$ what to find $M_{Z_n}(t)$, known ~~Mx~~ $E(X)$, $Var(X)$.

$Z_n = -\frac{\mu\sqrt{n}}{\sigma} + \frac{1}{\sqrt{n}\sigma} \cdot S_n$ [• S_n relates to X_i • what to apply MGF property ①]

$M_{Z_n}(t) = e^{-\frac{\sqrt{n}\mu}{\sigma} \cdot t} M_{S_n}\left(\frac{1}{\sqrt{n}\sigma} \cdot t\right)$ } $S_n = \sum_{i=1}^{n} X_i$ sum of iid

$= e^{-\sqrt{n}\mu/\sigma} \cdot \left[M_X\left(\frac{1}{\sqrt{n}\sigma} \cdot t\right)\right]^n$ [MGF ^ Normal example]

want to show $M_{Z_n}(t) \to M_Z(t) = e^{\{\frac{1}{2}t^2\}}$; ~~equal~~ equivalently

$\log M_{Z_n}(t) = \log M_Z(t) \Rightarrow \underline{(-\sqrt{n}\mu/\sigma) \cdot t + n \log M_X\left(\frac{t}{\sqrt{n}\sigma}\right) = \frac{1}{2}t^2}$

Goal ☆

taylor expansion

$M_X(t)$ around 0.

$\therefore M_X\left(\frac{t}{\sqrt{n}\sigma}\right) = M_X(0) + \frac{t}{\sqrt{n}\sigma} \cdot M_X'(0) + \frac{1}{2!}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 M_X''(0) + \frac{1}{3!}\left(\frac{t}{\sqrt{n}\sigma}\right)^3 M_X'''(0) + \dots$

$= 1 + \frac{t}{\sqrt{n}\sigma} \cdot \mu + \frac{1}{2!}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 \cdot E(X^2) + \frac{1}{3!}\left(\frac{t}{\sqrt{n}\sigma}\right)^3 E(X^3) + \dots$ [$M_X(0) = E(e^{0X}) = E(1) = 1$]

taylor expansion $\log M_X(t)$

$\log M_X\left(\frac{t}{\sqrt{n}\sigma}\right) = \log\{1 + \underbrace{\frac{t}{\sqrt{n}\sigma} \cdot \mu + \frac{1}{2!}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 (\mu^2+\sigma^2) + \frac{1}{3!}\left(\frac{t}{\sqrt{n}\sigma}\right)^3 EX^3 + \dots}_{x}\}$

$\log(1+x) = \log(1+0) + x \cdot \frac{d\log(1+x)}{dx}\Big|_{x=0} + \frac{1}{2!}x^2 \cdot \frac{d^2\log(1+x)}{dx^2}\Big|_{x=0} + \dots$

$= 0 + x \cdot \left(\frac{1}{1+x}\Big|_{x=0}\right) + \frac{1}{2!}x^2\left(\underbrace{-\frac{1}{(1+x)^2}\Big|_{x=0}}_{-1}\right)$

$= 0 + x - \frac{1}{2}x^2 + \dots$

$\therefore \log(1+x) \approx x$

~~$\Rightarrow \log M_X\left(\frac{t}{\sqrt{n}\sigma}\right) = \log\{1+x\} \approx x = \frac{t}{\sqrt{n}\sigma} \cdot \mu + \frac{1}{2!}\left(\frac{t}{\sqrt{n}\sigma}\right)^2(\mu^2+\sigma^2) + \frac{1}{3!}\left(\frac{t}{\sqrt{n}\sigma}\right)^3 EX^3 \dots$~~

~~$= \frac{t}{\sqrt{n}\sigma} \cdot \mu + \frac{1}{2}\frac{t^2}{n\sigma^2}(\mu^2+\sigma^2) + o\left(\frac{1}{n}\right)$~~ $\frac{1}{n\sqrt{n}} = o\left(\frac{1}{n}\right)$ ↳ ~~faster than $\frac{1}{n}$~~

$\log M_X\left(\frac{t}{\sqrt{n}\sigma}\right) = \log\{1+x\} = 0 + x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3$

$= \underbrace{\boxed{\frac{t}{\sqrt{n}\sigma}\mu + \frac{1}{2}\frac{t^2}{n\sigma^2}(\mu^2+\sigma^2) + o\left(\frac{1}{n}\right)}}_{x}$ —② → [x^2 → open each term, see it each term converge to 0 faster than $\frac{1}{n}$]

$-\frac{1}{2}\left(\frac{t^2}{n\sigma^2}\mu^2 + o\left(\frac{1}{n}\right)\right)$ Small o: relation between a_n, b_n

$= \frac{t}{\sqrt{n}\sigma}\mu + \frac{1}{2}\frac{t^2}{n\sigma^2} \cdot \sigma^2 + o\left(\frac{1}{n}\right)$

$= \frac{t}{\sqrt{n}\sigma}\mu + \frac{1}{2}\frac{t^2}{n} + o\left(\frac{1}{n}\right)$

back to our goal: $(-\sqrt{n}\mu/\sigma)t + n\log M_X\left(\frac{t}{\sqrt{n}\sigma}\right) = -\sqrt{n}\mu/\sigma + \frac{\sqrt{n}t}{\sigma}\mu + \frac{t^2}{2} + \underbrace{o(1)}_{\to 0} = \frac{t^2}{2}$

~~$= \frac{t^2}{2}$~~

$\therefore \log M_{Z_n}(t) \xrightarrow{n\to\infty} \frac{1}{2}t^2 \Rightarrow M_{Z_n}(t) \longrightarrow M_Z(t) = e^{-\frac{1}{2}t^2}$

$\Rightarrow$ By continuity thm, $Z_n \to Z \Rightarrow Z_n \xrightarrow{d} N(0,1)$