PDF · $P(a \le X \le b) = \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

↳ argument symbol doesn't matter.

- $f(x) \ge 0$
- $\int_{\mathcal{X}} f(x)\,dx = 1$

$= E(\boxed{I(a<X<b)})$

R.V. $I(a<X<b) \sim \text{Bernulli}(p(a<X<b))$

expectation = p for Bernulli

CDF $P(X \le x) = \int_{-\infty}^{x} f(y)\,dy$

↳ PDF at point y, any y values in $(-\infty, x)$

Here, x is a fixed value, so can't use $\int_{-\infty}^{x} f(x)\,dx$, we need a symbol to represent whatever value of $f(\cdot)$.

moments 1^{st} moment = mean = expectation $= E[X] = EX = \begin{cases} \sum_{x \in \mathcal{X}} x\,p(X=x) \\ \int_{\mathcal{X}} x \cdot f(x)\,dx \end{cases}$

2^{nd} moment $= EX^2 = \begin{cases} \sum_{\mathcal{X}} x^2 p(X=x) \\ \int_{\mathcal{X}} x^2 f(x)\,dx \end{cases}$

alway ask "what's the R.V."

MGF X. R.V. $\in \mathbb{R}$, $\forall t \in \mathbb{R}$, def MGF of X is.

$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} p(X=x) \\ \int_{-\infty}^{+\infty} e^{tx} f(x)\,dx \end{cases}$

Taylor expression $e^{tx} = 1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!}$ on exponential function e^{tx}.

① $M_X(t) = E(e^{tx}) = 1 + \frac{tEX}{1!} + \frac{t^2EX^2}{2!} + \frac{t^3EX^3}{3!}$

② $M_X(t) = M_X(0) + \frac{tM_X'(0)}{1!} + \frac{t^2M_X''(0)}{2!} + \frac{t^3M_X'''(0)}{3!} + \ldots$

① = ② $M_X(0) = E(e^{0x}) = E(1) = 1$

$\frac{tM_X'(0)}{1!} = \frac{tEX}{1!} \Rightarrow \boxed{M_X'(0) = EX}$

$\frac{t^2M_X''(0)}{2!} = \frac{t^2EX^2}{2!} \Rightarrow \boxed{M_X''(0) = EX^2}$

① = ② works as long as Taylor expansion for $M_X(t)$ works around 0.

$M_X(t)$ continuous and smooth around zero.

$\therefore E(X^k) = M_X^{(k)}(0)$ if $M_X^{(k)}(t)$ exist at an interval around 0.

binomial MGF

$\cancel{M_X(t)} = p(X=x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, 2 \cdots n$

$M_X(t) = E(e^{tX}) = \sum_{x \in X} e^{tx} \cdot p(X=x)$

$= \sum_{x \in X} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$

$= \sum_{x=0}^{n} \binom{n}{x} (e^t \cdot p)^x (1-p)^{n-x} \Rightarrow$ binomial ~~for~~ formula.

$= (e^t p + 1 - p)^n$

binomial formula

$(a+b)^n = \cancel{ab^0 + a^2 b^{n-2} + a^3 b^{n-3} + \cdots a}$

$= \cancel{a^n b^0 + a^{n-1} b^1 + a} \binom{n}{n} a^n b^0 + \binom{n}{n-1} a^{n-1} b^1 + \binom{n}{n-2} a^{n-2} b^2$

$+ \cdots \binom{n}{1} a b^{n-1} + \binom{n}{0} a^0 b^n$

$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$M_X'(t) = n(e^t p + 1 - p)^{n-1} \cdot p \cdot e^t \Rightarrow M_X'(0) = n \cdot p = EX$

$M_X''(t) = n \cdot (n-1)(e^t p + 1 - p)^{n-2} e^t p \cdot p e^t + n(e^t p + 1 - p)^{n-1} \cdot p \cdot e^t$

$M_X''(0) = n(n-1) \cdot p^2 + n \cdot p = n^2 p^2 - np^2 + np \cancel{-n^2 p}$

poisson MGF

$X \sim \text{poisson}(\lambda) \; \lambda > 0 \quad P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$ $\left\{ \begin{array}{l} \sum_{x=0}^{\infty} P(X=x) = 1 \\ \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = 1 \end{array} \right.$

$M_X(t) = \cancel{E} E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} P(X=x)$

using the fact that $\sum_{x=0}^{\infty} P(X=x) = 1$, $\sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = 1$

$= \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!}$

$= \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x \cdot e^{-e^t \lambda}}{x!} \cdot \frac{e^{-\lambda}}{e^{-e^t \lambda}}$

1

$= \cancel{\sum_{x=0}^{\infty} \frac{e^{-\lambda}}{e^{-e^t \lambda}}} = \cancel{\infty}$

$= \frac{e^{-\lambda}}{e^{-e^t \lambda}} \cdot \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x \cdot e^{-e^t \lambda}}{x!} = e^{-\lambda + e^t \lambda}$

$\therefore M_X'(t) = e^{-\lambda + e^t \lambda} \cdot e^t \lambda \quad M_X'(0) = \lambda$

$M_X''(t) = (e^{-\lambda + e^t \lambda}) \lambda \cdot e^t \lambda + e^{-\lambda + e^t \lambda} \cdot \lambda \Rightarrow M_X''(0) = \lambda^2 + \lambda$ ✓

$EX^2 = \text{var}X + (EX)^2 = \lambda + \lambda^2$ ✓

exponential MGF

$X \sim Exp(\lambda)$ $\lambda > 0$. PDF $f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$

$E(e^{tX}) = \int_0^{\infty} e^{tx} \cdot f(x)\,dx$ ▷ using the fact that

$= \int_0^{\infty} e^{tx} \cdot \lambda e^{-\lambda x}\,dx$ $\qquad \int_0^{\infty} \lambda \cdot e^{-\lambda x}\,dx = 1 \quad \forall \lambda > 0.$

$= \int_0^{\infty} \lambda \cdot e^{-(\lambda - t)x}\,dx$

$= \int_0^{\infty} \frac{\lambda(\lambda - t) \cdot e^{-(\lambda - t)x}}{\lambda - t}\,dx$ [restore the original term]

$= \frac{\lambda}{\lambda - t}$ [if $\lambda - t > 0$]

↳ we only want to evaluate $M_X(0)$

Standard Normal MGF

$X \sim N(0,1)$ $f(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}$.

$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{\{-\frac{x^2}{2}\}}\,dx$ ▷ using the fact that $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\} = 1$

$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\{-(\frac{x^2}{2} - tx)\}}\,dx$

$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\{-\frac{(x^2 - 2tx)}{2}\}}\,dx$ $\qquad x^2 - 2tx + t^2$

$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\{-\frac{\{(x-t)^2 - t^2\}}{2}\}}\,dx$

$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} \cdot e^{\frac{t^2}{2}}\,dx$

$= e^{\frac{t^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}}\,d(x-t)$ shift x by t.

$= e^{\frac{t^2}{2}}$

properties of MGF

① If X has MGF $M_X(t)$ and $Y = a + bX$, $a, b \in \mathbb{R}$.
Then Y has MGF $M_Y(t) = e^{at} \cdot M_X(bt)$.

Pf: ~~E~~ $M_Y(t) = E(e^{Yt}) = E(e^{(a+bX)t}) = E(e^{at} \cdot e^{bXt}) = e^{at} \underline{E(e^{btX})}$
$= e^{at} \underline{M_X(bt)}$ [by defn] $\left\{ \begin{array}{l} E(e^{tX}) = M_X(t) \\ E(e^{btX}) = M_X(bt) \end{array} \right.$

Example: [using property ①] $Y \sim N(\mu, \sigma^2)$ $Y = \mu + \sigma X$ where $X \sim N(0,1)$.

$\therefore M_Y(t) = e^{\mu t} \cdot M_X(\sigma t)$ [we know $M_X(t) = e^{\frac{t^2}{2}}$]
$= e^{\mu t} \cdot e^{\frac{(\sigma t)^2}{2}} = e^{\mu t + \frac{(\sigma t)^2}{2}}$

[$Y \sim N(\mu, \sigma^2)$ is complex, $X \sim N(0,1)$ PDF is neat, so we can use MGF property ① to get the MGF of Y]

② X and Y are indep R.V.s with $M_X(t)$ and $M_Y(t)$ respectively, Then the MGF of $X+Y$ is
$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

Recall indep $P(X \in \mathcal{X}, Y \in \mathcal{Y}) = P(X \in \mathcal{X})\, P(Y \in \mathcal{Y})$
$P(X \in \mathcal{X} \mid Y \in \mathcal{Y}) = P(X \in \mathcal{X})$

Pf $M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX} \cdot e^{tY}) \overset{\text{indep}}{=} E(e^{tX})\, E(e^{tY}) = M_X(t) M_Y(t)$

Example [using property ②]
- if $X \sim \text{Binomial}(n_1, p)$, $Y \sim \text{Binomial}(n_2, p)$, and X, Y are indep.

~~Ete~~ $M_X(t) = (e^t p + 1 - p)^{n_1}$ $M_Y(t) = (e^t p + 1 - p)^{n_2}$ same base
then $M_{X+Y}(t) = (e^t p + 1 - p)^{n_1} \cdot (e^t p + 1 - p)^{n_2} = \underline{(e^t p + 1 - p)}^{n_1 + n_2}$
$\Rightarrow M_{X+Y}(t)$ has form of binomial, so $X + Y \sim \text{Bin}(n_1 + n_2, p)$

- $X \sim \text{pois}(\lambda_1)$, $Y \sim \text{pois}(\lambda_2)$ X, Y indep. λ_1, λ_2: # of events.

~~E~~ $M_X(t) = e^{\lambda_1(e^t - 1)}$ $M_Y(t) = e^{\lambda_2(e^t - 1)}$ $M_{X+Y}(t) = e^{(\lambda_1 + \lambda_2)(e^t - 1)}$

$M_{X+Y}(t)$ ~~ha~~ belongs to the form of a pois. dist. ~~AA~~ $X+Y \sim \text{pois}(\lambda_1 + \lambda_2)$

intuition: # of events in restaurant 1 + # of events in restraunt 2
= total # of events in rest 1 & 2.

example ▸ $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, X & Y indep.

$M_X(t) = e^{\{\mu_1 t + \frac{\sigma_1^2 t^2}{2}\}}$ $M_Y(t) = e^{\{\mu_2 t + \frac{\sigma_2^2 t^2}{2}\}}$

$M_{X+Y}(t) = e^{\{\mu_1 t + \frac{\sigma_1^2 t^2}{2}\}} \cdot e^{\{\mu_2 t + \frac{\sigma_2^2 t^2}{2}\}} = e^{\{(\mu_1+\mu_2)t + \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}\}}$

$\therefore X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

- Without MGF, we already know $E(X+Y) = E(X)+E(Y)$ ↳ Holds for any X & Y, doesn't require indep.
  $Var(X+Y) = Var(X) + Var(Y)$ [X, Y indep]
- MGF confirms $X+Y \sim$ Normal.

▸ $X_1, X_2 \dots X_n \overset{iid}{\sim}$ R.V.s. what's the MGF of $\sum_{i=1}^{n} X_i$?

$M_{\sum_{i=1}^n X_i}(t) = E(e^{t \cdot \sum X_i}) = E(e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n})$

$\overset{indep}{=} E(e^{tX_1}) E(e^{tX_2}) \cdots E(e^{tX_n}) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t)$

$\overset{\text{identical dist}}{=} [M_X(t)]^n$

$X_1, \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ $M_{\sum X_i}(t) = \left[e^{\{\mu t + \frac{\sigma^2 t^2}{2}\}}\right]^n = \left\{e^{\mu t} \cdot e^{\frac{\sigma^2 t^2}{2}}\right\}^n$

$= e^{n\mu t} \cdot e^{\frac{n\sigma^2 t^2}{2}}$ ↳ nice property of exp family, put n into the power

$\therefore \sum X_i \sim N(n\mu, n\sigma^2)$

Joint MGF consider a pair of R.V.s (X, Y) $X, Y \in \mathbb{R}$

$M_{X,Y}(s,t) = E(e^{sX+tY}) = E(e^{(s,t)^T \binom{X}{Y}})$ ↳ inner product. $s, t \in \mathbb{R}$

if X, Y discrete $M_{X,Y}(s,t) = \sum_y \sum_x e^{sx+ty} P(X=x, Y=y)$

continuous

$M_{X,Y}(s,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{sx+ty} f(x,y) \, dx dy$

If X, Y are indep, we have:

$M_{X,Y}(s,t) = E(e^{sX+tY}) = E(e^{sX} \cdot e^{tY}) \overset{indep}{=} E(e^{sX}) E(e^{tY})$

↳ to obtain the joint moments of (X, Y) $= M_X(s) M_Y(t)$

$E(X^r Y^v) = \left.\frac{\partial^{r+v} M_{X,Y}(s,t)}{\partial s^r \partial t^v}\right|_{s=0, t=0}$ [use two-dimension taylor expansion]

(r, v)th moment

properties ① $M_{X,Y}(s, 0) = E(e^{sX}) = M_X(s)$

② $M_{X,Y}(0, t) = E(e^{tY}) = M_Y(t)$

③ $M_{X,Y}(t,t) = E(e^{tX+tY}) = E(e^{t(X+Y)}) = M_{X+Y}(t)$