marginal prior of $u \sim \cancel{t_{2\alpha}}\ t_{2\alpha}$ location $= u_0$, precision $= \frac{\alpha\theta}{\beta}$

marginal posterior of $u \sim t_{2\alpha'}$ location $= u_0'$ precision $= \frac{\alpha'\theta}{\beta'}$

So, when u, τ both are unknown, we specify the following priors parameters: $u_0, \theta, \alpha, \beta$.

joint posterior dist of τ & u. $\tau \sim \text{Gamma}(\alpha', \beta')$ $u|\tau \sim N(u_0', \theta\tau_0')$

- post ~~prec~~ precision $(\theta + n)\cdot\tau$, $n\uparrow$. precision $\uparrow$
- $\alpha' = \alpha + n/2$ $n\uparrow$. $\alpha'\uparrow$, Gamma dist gets "tighter"

No-info prior: posterior $u|\theta\ \tau \sim N(\cancel{\alpha,\beta}\ u_0', \tau_0')$

$\hookrightarrow u_0 =$ some value, $\theta = 0$, $\alpha = 0$, $\beta = 0$

$u_0' = \bar{x}$, $\tau_0' = n\tau$

$\alpha' = n/2$, $\beta' = \Sigma(x_i - \bar{x})^2/2$. $E(\frac{1}{\tau}) = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$

marginal post of $u \sim t_n / t_{2\alpha'}$

~~assi~~

- both u, τ unknown.
- start with $u^{(0)}$, $\tau^{(0)}$ say $(u^{(0)}, \tau^{(0)}) = (20, 0.0025)$

  note: $u^{(0)}, \tau^{(0)}$ are params of x
- first we assume τ is known [i.e., the precision of x dist known which is $\tau^{(0)}$]

  given $\tau = \tau^{(0)}$, we can calculate post ~~dist~~ of $u|u_0', \tau_0'$

  $\tau_0' = (\theta + n)t = (4+6)\cdot\tau^{(0)} = 10 \cdot 0.0025 = 0.025$

  $u_0' = \frac{\theta u_0 + n\bar{x}}{\theta + n} = \frac{4\times 20 + 6\times 67.33}{10} =$

  till now, we $\Rightarrow$ get $u|u_0', \tau_0' \sim N(u_0', \tau_0')$
- update param of x: $u^{(1)}$. by drawing a sample from post $u|\tau = \tau^{(0)} \sim N(u_0', \tau_0')$

  $u^{(1)} = 61.417$.
- given $u = u^{(1)}$, $\alpha' = \alpha + \frac{n+1}{2}$, $\beta' = \beta + \frac{1}{2}(\Sigma(x_i - u)^2 + \theta(u - u_0)^2)$

Markov chain Monte carlo method

- specify the model:

$$X_i \mid \mu, \tau \sim N(\mu, \tau)$$
$$\mu \mid \mu_0, \theta, \tau \sim N(\mu_0, \theta\tau)$$
$$\tau \mid \alpha, \beta \sim \text{Gamma}(\alpha, \beta)$$

- specify prior params: $\mu_0, \theta, \alpha, \beta$.

$\mu_0 = 66, \ \theta = 4, \ \alpha = 1, \ \beta = 25.$

- the data: $\bar{X} = 67.333, \ \Sigma(X_i - \bar{X})^2 = 88.33$

↳ Assume τ is ~~kown~~ known. then prior of μ is fully specified. posterior μ can be calculated.

$\mu_0' = \frac{\theta\mu_0 + n\bar{X}}{\theta + n} = 66.80 \quad \tau_0' = (\theta + n)\tau = (4+6)\tau = 10\tau$ [assume τ known]

post $\mu \mid \tau \sim (\mu_0', \tau_0')$

↳ Assume μ is known

Posterior $\tau \sim \text{Gamma}(\alpha', \beta')$, where

$\alpha' = \alpha + \frac{n+1}{2} \qquad \beta' = \beta + \frac{1}{2}\left(\Sigma(X_i - \mu)^2 + \theta(\mu - \mu_0)^2\right)$ [assume μ known]

- sample

start with $\mu^{(0)}, \tau^{(0)} = (20, 0.0025)$

↳ assume $\tau = \tau^{(0)}$ known, then we can get the posterior $\mu \mid \mu_0', \tau_0'$

$\mu_0' = \frac{\theta\mu_0 + n\bar{x}}{\theta + n} = \frac{4 \cdot 66 + 6 \cdot 67.33}{4+6} = 66.80 \qquad \tau_0' = (\theta + n)\tau = (4+6)\tau_0 = 10 \cdot 0.0025 = 0.025$

Now we know μ is from posterior μ dist $N(\mu_0', \tau_0')$

↳ sample one μ from post $\mu \mid \mu_0', \tau_0' \sim N(\mu_0', \tau_0')$ → in a sense $\mu^{(0)}$ is been updated, the sampled μ value is $\mu^{(1)} = 61.417$

↳ As we known $\mu = \mu^{(1)}$ [latest μ], assume $\mu = \mu^{(1)}$ is true/known, we can computed the posterior τ ~~$\sim t\alpha'$~~, $\sim \text{Gamma}(\alpha', \beta')$, where $\alpha' = \alpha + \frac{n+1}{2}$, $\beta' = \beta + \frac{1}{2}\left(\Sigma(x_i - \mu)^2 + \theta(\mu - \mu_0)^2\right)$, μ assumed known $= \mu^{(1)}$.

sample one τ from post $\tau \sim \text{Gamma}(\alpha', \beta')$, to set $\tau^{(1)}$

↳ till now, we've sot a pair of now param values for X: $\mu^{(1)}, \tau^{(1)}$

we repeat the before-mentioned process.
Assume τ known $= \tau^{(n-1)}$ → able to compute post $\mu \sim (\mu_0', \tau_0')$
→ sample one value from post μ to set $\mu^{(n)}$ → use $\mu^{(n)}$ as true μ [i.e., assume μ known with value $= \mu^{(n)}$] → able to calculate post τ follows $\Gamma(\alpha', \beta')$ → sample one from post $\Gamma(\alpha', \beta')$ to set $\tau^{(n)}$

state 1: $(\mu^{(0)}, \tau^{(0)})$ → compute post μ → sample → state 2: $(\mu^{(1)}, \tau^{(1)})$ → compute post τ → sample → compute post μ of X ☆ → sample → state 3: $(\mu^{(2)}, \tau^{(2)})$ → compute post τ → sample ...... state m: $(\mu^{(m)}, \tau^{(m)})$

- each state is a dist^n summarized by $\mu^{(i)}, \tau^{(i)}$
- Markov chain is created to obtain samples from posterior dist^s of params of interest.

- marginal density of μ: $\mu^{(0)}, \mu^{(1)}, \mu^{(2)} \ldots \mu^{(m)}$
  marginal density of τ: $\tau^{(0)}, \ldots \tau^{(m)}$
  joint density (μ, τ): $(\mu^{(0)}, \tau^{(0)}) \ldots (\mu^{(m)}, \tau^{(m)})$

Gibbs sampler (one of the oledest MCMC algorithms)

Suppose we have $\bar{\theta} = (\theta_1, \theta_2, \theta_3)$ in the Bayesian model
Given m-th updated values $\theta = \theta_1^{(m)}, \theta_2^{(m)}, \theta_3^{(m)}$, the $m+1$th sample is:

$$\theta_1^{(m+1)} \sim [\theta_1 \mid \theta_2 = \theta_2^{(m)}, \theta_3 = \theta_3^{(m)}]$$
$$\theta_2^{m+1} \sim [\theta_2 \mid \theta_1 = \theta_1^{(m+1)}, \theta_3 = \theta_3^{(m)}]$$
$$\theta_3^{m+1} \sim [\theta_3 \mid \theta_1 = \theta_1^{(m+1)}, \theta_2 = \theta_2^{m+1}]$$
$$\frac{1}{M}\sum_{m=1}^{M} g(\theta^{(m)}) \xrightarrow{a.s.} E(g(\theta) \mid \text{data})$$

~~e~~-way anova model

- the data: Y_{ij} ~~jth~~ subject j oxygen measurement in lab condition i.

  i=1 : 7 9 5 5 10 11 ....

  i=2 : 15 11 15 17 12 ...

  i=3 : 11 10 15 4 8 ...

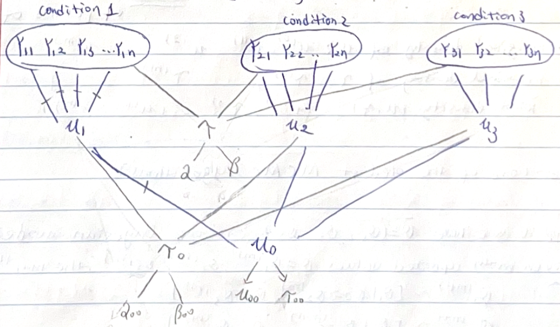
- Specify the model [conditional PDF]

$$Y_{ij} \mid u_i, \tau \sim N(u_i, \tau) \quad \text{[for each lab]}$$
$$u_i \mid u_0, \tau_0 \sim N(u_0, \tau_0)$$
$$u_0 \mid u_{00}, \tau_{00} \sim N(u_{00}=10, \tau_{00}=0.001)$$
$$\tau \mid \alpha, \beta \sim \text{Gamma}(\alpha=0.01, \beta=0.01)$$
$$\tau_0 \mid \alpha_0, \beta_0 \sim \text{Gamma}(\alpha_0=0.01, \beta_0=0.01)$$

using Gibbs sampler, ~~assu~~ first assume all params are known [i.e, starting values] ⇒ calcute the conditional dist of each param

What we have: Y_{ij}, u_i, u_0, τ, τ_0, u_{00}, τ_{00}, α_0, β_0

conditional PDF of post u_1 : $f(u_1 \mid Y_{ij}, u_i, u_0, \tau, \tau_0, \cancel{\alpha_0} u_{00}, \tau_{00}, \alpha_0, \beta_0)$



u_1 only related to $\underline{Y_{11}\ Y_{12} \ldots Y_{1n}}$ u_0, τ_0, τ.

post of μ_i ∴ the conditional pdf can be reduced to $f(\mu_1 | Y_{1j}, \tau, \mu_0, \tau_0)$.

- prior $\mu_1 \sim N(\mu_0, \tau_0)$; specify μ_0, τ_0.
- $Y_{1j} \sim N(\mu_1, \tau)$ assume τ is known.

} ⇒ reduced to simpler case encountered before

- posterior of $\mu_1 \sim N(\mu_{01}', \tau_{01}')$, where

$$\mu_{01}' = \frac{\tau_0 \mu_0 + n_1 \tau \bar{Y}_1}{\tau_0 + n_1 \tau}$$

post of τ Given Y_{ij}, μ_i, post dist of τ is Gamma with

$$\alpha' = \alpha + (n_1 + n_2 + n_3)/2 \qquad \beta' = \beta + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{n_i} (Y_{ij} - \mu_i)^2$$

Note: post of μ_i, τ caculation: refer to Normal case notes.

post of μ_0 ∴ The data: μ_1, μ_2, μ_3; $n = 3$, $\bar{x} = \frac{\mu_1 + \mu_2 + \mu_3}{3}$

- The model: $\mu_i \sim N(\mu_0, \tau_0)$.
- The prior: $\mu_0 \sim N(\mu_{00}, \tau_{00})$, specify μ_{00}, τ_{00} values.
- The post of $\mu_0 \sim N(\mu_{00}', \tau_{00}')$

$$\mu_{00}' = \frac{\tau_0 \mu_0 + n \tau \bar{x}}{\tau_0 + n\tau} = \frac{\tau_{00} \mu_{00} + 3 \cdot \tau_0 \cdot \bar{x}}{\tau_{00} + 3 \cdot \tau_0}$$

↳ normal case, symbols are placeholders. change correspondingly

τ_0: replace with prior precision for μ_0

μ_0: replace with ~~pt~~ prior mean for μ_0

τ: replace with precision for μ_i.

post of τ_0 · ~~α~~ prior of $\tau_0 \sim \Gamma(\alpha_{0p}, \beta_{0p})$ then Given μ_i, post $\tau_0 \sim \Gamma(\alpha_{00}', \beta_{00}')$

$$\alpha_{0p}' = \alpha_{0p} + \frac{n}{2} = \alpha_{0p} + \frac{3}{2} \qquad \beta_{0p}' = \beta_{0p} + \frac{1}{2} \sum_{i=1}^{3} (\mu_i - \mu_0)^2$$

Takeaway: break up to pieces. reduce to simplier normal case

- post of μ_i, τ, μ_0, τ_0 are all conditional PDFs.