Sum of two indep random variables.

$U_1 \sim Unif(0,1) \qquad U_2 \sim Unif(0,1)$

$Z = \frac{U_1+U_2}{2}$, what's the PDF of Z. : solution: $f_Z(z) = \begin{cases} 4z & 0<z< \\ 4-4z & \frac{1}{2}<z \end{cases}$

$\frac{U_1}{2} \sim Unif(0, \frac{1}{2})$, $\frac{U_2}{2} \sim Unif(0, \frac{1}{2})$

∴ the joint PDF of $\frac{U_1}{2}$, and $\frac{U_2}{2}$ is

$$f(\frac{U_1}{2}, \frac{U_2}{2}) \overset{indep}{=} f(\frac{U_1}{2}) f(\frac{U_2}{2}) = \underbrace{\frac{1}{\frac{1}{2}-0}}_{Unif(0,\frac{1}{2})} \cdot \underbrace{\frac{1}{\frac{1}{2}-0}}_{Unif(0,\frac{1}{2})} = 4$$

∴ the PDF of Z is given by:

$$f_Z(\frac{U_1}{2}+\frac{U_2}{2}) = \int_{\frac{U_1}{2}} \underbrace{f(\frac{U_1}{2}, z-\frac{U_1}{2})}_{\text{joint PDF of } \frac{U_1}{2} \,\&\, \frac{U_2}{2}} d\frac{U_1}{2}$$ all possible combinations of $\frac{U_1}{2}$

= two parts when apply change of var

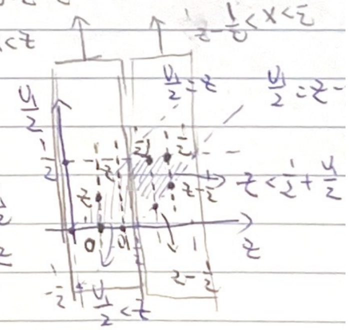
$\left.\begin{array}{l} 0<\frac{U_1}{2}<\frac{1}{2} \\ 0<\underbrace{z-\frac{U_1}{2}}_{\frac{U_2}{2}}<\frac{1}{2} \end{array}\right\} \Rightarrow$ $\frac{U_1}{2} \to z$: $0<\frac{U_1}{2}<\frac{1}{2}$, $\frac{U_1}{2}<z<\frac{1}{2}+\frac{U_1}{2}$ ← see black dots ←

$x = \frac{U_1}{2}$

$\int$ over $\frac{U_1}{2}$ ⇒ ∴ y-axis: $\frac{U_1}{2}$, x-axis: z

① when $0<z<\frac{1}{2}$: $0<x<z$

② when $\frac{1}{2}<z<1$: $z-\frac{1}{2}<x<\frac{1}{2}$



$$∴ f_Z(z=\frac{U_1}{2}+\frac{U_2}{2}) = \int_0^z f(\frac{U_1}{2}, z-\frac{U_1}{2})\, d\frac{U_1}{2} \quad \text{when } 0<z<\frac{1}{2}$$

$$= \boxed{4\Big|_0^z = 4z} \quad \text{when } 0<z<\frac{1}{2}$$

$$= \int_{z-\frac{1}{2}}^{\frac{1}{2}} f(\frac{U_1}{2}, z-\frac{U_1}{2})\, d\frac{U_1}{2} \quad \text{when } \frac{1}{2}<z<1$$

$$= 4\Big|_{z-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{2}\cdot 4 - (z-\frac{1}{2})\cdot 4 = 2-4z+2 = \boxed{4-4z} \quad \text{when } \frac{1}{2}<z$$