Normal sample variance

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写成这样非常重要，更加 intuitive.

记 $\dfrac{(n-1)s^2}{\sigma^2} = \dfrac{\Sigma(x_i-\bar{x})}{\sigma^2} = \Sigma\left(\dfrac{x_i-\bar{x}}{\sigma}\right)^2 \sim \chi^2(n-1)$?

证明思路：$X_i \sim N(\mu, \sigma^2)$ $\qquad S^2 = \dfrac{1}{n-1}\Sigma(x_i-\bar{x})$

$x^2 \sim \chi^2(1)$

$\left(\dfrac{x_i-\mu}{\sigma}\right)^2 \sim \chi^2(n)$

☆ $x_i-\mu$ 可以写为两部分 $\Sigma\left(\dfrac{x_i-\bar{x}_n+\bar{x}_n-\mu}{\sigma}\right)^2$

$$\underbrace{\Sigma\left(\frac{x_i-\mu}{\sigma}\right)^2}_{W} = \sum_{i=1}^{n}\left(\frac{x_i-\bar{x}_n}{\sigma} + \frac{\bar{x}_n-\mu}{\sigma}\right)^2$$

$$= \sum_{i=1}^{n}\left(\frac{x_i-\bar{x}_n}{\sigma}\right)^2 + n\left(\frac{\bar{x}_n-\mu}{\sigma}\right)^2$$

$$+ 2\,\frac{\bar{x}_n-\mu}{\sigma}\cdot\sum_{i=1}^{n}\frac{x_i-\bar{x}_n}{\sigma}$$

$\to$ well-known $\Sigma(x_i-\bar{x}_n) = 0$

$$= \sum_{i=1}^{n}\left(\frac{x_i-\bar{x}_n}{\sigma}\right)^2 + n\left(\frac{\bar{x}_n-\mu}{\sigma}\right)^2$$

$\to$ R.V. $U \perp\!\!\!\perp V$

of interest $\quad U \qquad V \qquad S^2 \perp\!\!\!\perp \bar{X}$

$W = U + V. \xrightarrow{U \perp\!\!\!\perp V} M_W(t) = M_U(t) M_V(t)$

$W = \sum \left(\frac{X_i - \mu}{\sigma}\right)^2$ 其 $\sim \chi^2(n)$

$$\left.\begin{array}{l} \frac{X_i - \mu}{\sigma} \sim N(0,1) \\ \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2(1) \end{array}\right\} \Rightarrow \sum \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \underline{\chi^2(n) \quad W!}$$

$V = n\left(\frac{\bar{X}_n - \mu}{\sigma}\right)^2$, $\quad \bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$ [用MGF可证]

$= \left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\right)^2 \qquad \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0, 1)$

$$V = \left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\right)^2 \sim \underline{\chi^2(1) \quad V!}$$

Gamma MGF: $\left(\frac{1}{1 - \beta t}\right)^{\alpha}$

$W \sim \chi^2(n) = \Gamma(\frac{n}{2}, 2)$, $V \sim \chi^2(1) = \Gamma(\frac{1}{2}, 2)$

$$M_U(t) = \frac{M_W(t)}{M_V(t)} = \frac{\left(\frac{1}{1-2t}\right)^{\frac{n}{2}}}{\left(\frac{1}{1-2t}\right)^{\frac{1}{2}}} = \left(\frac{1}{1-2t}\right)^{\frac{n-1}{2}}$$

$\Rightarrow$ we recongnize that $U \sim$ Gamma$\left(\frac{n-1}{2}, 2\right)$

$$= \chi^2(n-1) \quad \blacksquare$$