Tdist

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$X \sim N(\mu, \sigma^2)$

$U = \frac{X-\mu}{\sigma} \sim N(0,1)$

$t = \frac{U}{\sqrt{V/p}}$, $W = V$.

$U \to N(0,1)$, $V \to \chi^2(p)$

$t = \frac{U}{\sqrt{V/p}}$ $U \sim N(0,1)$

$V \sim \chi^2(p)$, U and V indep.

$\hookrightarrow \text{Gamma}(\frac{p}{2}, 2)$

$\alpha = \frac{p}{2}$ $\beta = 2$

$$f_{U,V}(u,v) = f_U(u) f_V(v)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}u^2\} \cdot \frac{1}{\Gamma(\frac{p}{2}) 2^{p/2}} v^{(\frac{p}{2})-1} e^{\frac{-v}{2}}$$

$$t = \frac{u}{\sqrt{v/p}} , \quad w = v$$

思路：知道 U. V. 以及他们的 PDF 求 $t = \frac{u}{\sqrt{v/p}}$ PDF

本质上就是用 change of variable.

$U = t\, w^{\frac{1}{2}} \cdot p^{-\frac{1}{2}}$ $\frac{du}{dw} = p^{-\frac{1}{2}} t \cdot \frac{1}{2} w^{-\frac{1}{2}}$ $\frac{du}{dt} = \sqrt{\frac{w}{p}}$

$v = w$

$$\frac{du}{dw} = 1 \qquad \frac{dw}{dt} = 0$$

$$\therefore |J| = \left| -\sqrt{\frac{w}{p}} \right| = \sqrt{\frac{w}{p}}$$

$$f_{T,W}(t,w) = f_U(u) f_V(v) \cdot |J|$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} t^2 \frac{w}{p}\right\} \cdot \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}}} \cdot w^{\frac{p}{2}-1} e^{-\frac{w}{2}} \; |J|$$

求 T 的 marginal PDF

$$f_T(t) = \int_{-\infty}^{+\infty} f_{T,W}(t,w) \sqrt{\frac{w}{p}} \, dw$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}}} \int \exp\left\{-\frac{1}{2} \frac{t^2}{p} \cdot w\right\} \cdot w^{\frac{p}{2}-1} \cdot e^{-\frac{w}{2}} \frac{w^{\frac{1}{2}}}{\sqrt{p}} \, dw.$$

这里需要合并.

$$= \frac{1}{\ } \frac{1}{\ } \int \exp\left\{-\frac{1}{2}\left(\frac{t^2}{p}+1\right) \cdot w\right\} \quad \frac{p}{2}-1+\frac{1}{2} \cdot p^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}}} \int \exp\{ - \cdots \} \, w \quad dw$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}} p^{\frac{1}{2}}} \int \exp\left\{-\frac{1}{2}\left(\frac{t^2}{p}+1\right) w\right\} \, w^{\frac{p}{2}-\frac{1}{2}} \, dw$$

妈呀，这是个 Gamma 的 kernel

$\exp\{-x/\beta\} \cdot x^{\alpha-1}$

$$\exp\left\{-w \Big/ \left[\frac{1}{2}\left(\frac{t^2}{p}+1\right)\right]^{-1}\right\} \cdot w^{\frac{p}{2}-\frac{1}{2}-1+1}$$

$$w^{\overbrace{\left(\frac{p}{2}+\frac{1}{2}\right)}^{\alpha}-1}$$

$$\beta = \left(\frac{\frac{t^2}{p}+1}{2}\right)^{-1}$$

$$= \Gamma(\alpha) \cdot \beta^{\alpha} = \Gamma\left(\frac{p}{2}+\frac{1}{2}\right) \cdot \left(\frac{2}{\frac{t^2}{p}+1}\right)^{\frac{p+1}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}} p^{\frac{1}{2}}} \Gamma\left(\frac{p}{2}+\frac{1}{2}\right) \cdot \left(\frac{2}{\cdots}\right)^{\frac{p+1}{2}}$$