Geometric and negative binomial

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首先，注意不同的 parametrization，这很重要，否则总是无法推导.

(SI appendix 中并不对应 ...)

① X: # failures

Geom(p) $\quad P(X=x \mid p) = p(1-p)^x \qquad x=0,1,2\ldots$

NB(p; r) $\quad P(Y=y \mid r,p) = p \cdot \binom{r+y-1}{y}(1-p)^y p^{r-1}$

$$= \binom{r+y-1}{y}(1-p)^y p^r \qquad y=0,1,2\ldots$$

① 总trail数量：成功+失败 $= r+x \qquad p^r(1-p)^x$

② 从 $r+x-1$ 中选出 x 个 failure.

Sidenote:

① 总trail数：成功+失败 $= r+x$

② 最后一次一定成功，在 $r+x-1$ 次中进行 bin 组合

$C(r+x-1, x)$ $r+x-1$ 次中，

$- - - - - - - - \mid r \rightarrow$ 最后一次一定成功

求一下 MGF 吧！

$$E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \cdot p(1-p)^x \qquad \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \Leftarrow \text{geom series}$$

$$= p\sum_{x=0}^{\infty} \underbrace{[e^t(1-p)]}_{r}{}^x \qquad r = e^t(1-p) \text{ common ratio } \underline{0<|r|<1}$$

$a=1 \quad a_1 = 1 \qquad e^t(1-p) < 1$

$$= \frac{p}{1-(1-p)e^t} \qquad S_\infty = \frac{1}{1-r} \qquad e^t < (1-p)^{-1}$$

$$\underline{t < -\log(1-p)}$$

constraint

$$E(e^{tY}) = \sum_{y=0}^{\infty} e^{ty}\binom{r+y-1}{y}\cdot(1-p)^y p^r$$

$$= p^r \sum_{y=0}^{\infty}\binom{r+y-1}{y}\underbrace{[(1-p)e^t]^y}_{-x=(1-p)e^t}$$

$$(x+a)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}x^k a^{-n-k}$$

取负号 $= \sum_{k}^{\infty}\binom{n+k-1}{k}(-x)^k a^{-n-k}$

$r+y-1$

$$= p^r(-(1-p)e^t+1)^{-r} \qquad x=-(1-p)e^t \quad a=1$$

$$= \left(\frac{p}{1-(1-p)e^t}\right)^r$$

② X: # of trails

Geom $\quad P(X=x \mid p) = p(1-p)^{x-1} \qquad x: 1,2,3\ldots$

NB(p; r) $\quad P(Y=y \mid r,p) = \binom{y-1}{r-1}p^r(1-p)^{y-r} \qquad y: 1,2,3\ldots$

求 MGF:

$$E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx}p(1-p)^{x-1}$$

$$= \frac{p}{1-p}\sum_{x=1}^{\infty}[e^t(1-p)]^x$$

$$= \frac{p}{1-p}\left(\sum_{x=0}^{\infty}[e^t(1-p)]^x - 1\right)$$

$$= \frac{p}{1-p}\left(\frac{1}{1-(1-p)e^t} - 1\right)$$

$$= \frac{p}{1-p}\cdot\frac{1-(1-(1-p)e^t)}{1-(1-p)e^t}$$

$$= \frac{p}{1-p}\cdot\frac{(1-p)e^t}{1-(1-p)e^t}$$

$$= \frac{pe^t}{1-(1-p)e^t}$$ 棒棒

$$E(e^{tY}) = \sum_{y=1}^{\infty} e^{ty}\cdot\binom{y-1}{r-1}p^r(1-p)^{y-r}$$

$$= \left(\frac{p}{1-p}\right)^r\sum_{y=1}^{\infty}\binom{y-1}{r-1}((1-p)e^t)^y$$

$y = r+k$ (success, failure)

$$= \left(\frac{p}{1-p}\right)^r\sum_{k=0}^{\infty}\binom{r+k-1}{r-1}((1-p)e^t)^{r+k}$$

$$= \left(\frac{p}{1-p}\right)^r\cdot((1-p)e^t)^r\cdot\underbrace{\sum_{k=0}^{\infty}\binom{r+k-1}{r-1}((1-p)e^t)^k}_{\text{negative bin series}} \rightarrow -x = (1-p)e^t$$

$x = -(1-p)e^t$

$a = 1$

$$= (pe^t)^r\cdot(1-(1-p)e^t)^{-r}$$

$$= \left[\frac{pe^t}{1-(1-p)e^t}\right]^r$$ 棒棒

∴ 几何分布的和为负二项分布 (p base)